# Nonlinear Instability of a Helicopter Blade in Hovering

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The nonlinear equations of flap-lag oscillations of a torsionally rigid blade are solved by a simplified asymptotic expansion procedure of multiple time scales. The region of stability and limit cycle oscillation and its comparison with numerical results are presented.

### I. Introduction

THE problem of the coupling between nonlinear oscillation, parametric excitation, and forced oscillation is of great interest in rotor dynamics. The present paper will restrict the considerations to the flap-lag motion. Few authors have considered the nonlinear oscillations of a torsionally rigid blade. Young<sup>1</sup> has used very crude approximations to derive the solution of a set of nonlinear equations. Hohenemser and Heaton<sup>2</sup> derived the nonlinear equations of motion in forward flight approximately and solved the equations by numerical iteration. The present author<sup>3</sup> used the equations derived by Hohenemser and applied the asymptotic expansion technique to establish amplitude of the limit cycle oscillation and clarified the role played by nonlinear oscillation, parametric excitation, and forced oscillation of the rotor blades. In the present paper, a more complete set of equations is studied. These equations are an extension of the linear equations in hovering derived by Ormiston and Hodge.<sup>4</sup> Similar techniques, as those of Ref. 3, are applied to these new equations. The case of hovering will be discussed in detail. Various cases in forward flight are given in Ref. 8. The expansion series using the present paper is somewhat different from those of Refs. 3 and 8 by taking advantage of the information on the magnitude of the coefficients of the equations.

## II. Formulation

In deriving the equations of motion, we shall make the following assumptions: 1) the elastic blade is approximated by a rigid blade hinged at a distance Re from the center of the hub with known spring constants in both the flap and the lag direction; 2) the flap moment is taken about the axis at the hinge-offset in the hub plane and normal to the instantaneous axis of the blade; 3) the lead-lag moment is taken about the axis parallel to the axis of rotation at the hinge-offset; 4) the blade is torsionally rigid and the feathering axis is the axis passing through the center of the hub and the hinge-offset; 5) the preconing,  $\beta_p$ , the prelagging,  $\zeta_p$ , and the pitch angle,  $\theta$ , are small so that their square can be neglected as compared to one; 6) the effective angle of attack is small and two-dimensional quasi-steady aero-

dynamics theory can be used; 7) the ratio of the profile drag coefficient to the lift slope is small and can be neglected as compared to one; and 8) nonlinear terms up to the third order will be retained. However, since the aerodynamic damping is much higher in flap than that in lag, the perturbed motion in lag is much larger than that in flap. It is expected that nonlinear terms of the perturbed lag motion  $\zeta$  will be more important than the perturbed flap motion  $\beta$ . It is also expected that the nonlinear terms are more important in the lag equation than in the flap equation. Thus the nonlinear terms such as  $\beta\zeta$ ,  $\beta\zeta^2$ ,  $\theta\zeta^2$  and their time derivatives will be retained in the flap equation while  $\beta^2$ ,  $\beta^2\zeta$ ,  $\theta\beta\zeta$ , etc. will be retained in the lag equations.

From the above assumptions, it can be shown that the aero-dynamic force in flap  $(L_z)_A$  and  $-(L_y)_{A'}$  [see Eq. (A17)], can be reduced to the more conventional expression, namely

$$(L_z)_A \cong \frac{1}{2}\rho_A \ ac \ U_T(U_T\theta - U_p)$$

$$(L_y)_A \cong \frac{1}{2}\rho_A \ ac \left[ U_p(U_p - U_T\theta) + \frac{C_{d_o}}{a} U_T^2 \right]$$
(1)

in which

$$\begin{split} U_{T} &\cong -V_{Y} + V_{Z} \theta \cong \Omega R \big[ \mu (\sin \psi + \zeta \cos \psi) - e - (1 - e) \xi (1 + \zeta) \big] \\ U_{p} &\cong -V_{Y} \theta - V_{Z} \\ &\cong \Omega R \big[ \lambda + \mu (\beta + \beta_{o}) (\cos \psi - \zeta \sin \psi) + (1 - e) \xi \beta \big] \end{split} \tag{2}$$

The equations of motion can now be simplified. We first obtain the static equation position  $\beta_o$ ,  $\zeta_o$  in hovering by setting the advanced ratio  $\mu$  and all the time derivatives to be zero in (A21) and (A22). We have

$$(1+I_{e})\beta_{o} + \frac{\omega_{\beta}^{2}}{\Omega^{2}}(\beta_{o} - \beta_{p}) = \frac{\gamma'}{8} \left[\theta + \frac{8}{3}e'\theta - A' - e'c\right]$$

$$\left(I_{e} - \frac{\omega_{\zeta}^{2}}{\Omega^{2}}\right)(\zeta_{o} - \zeta_{p}) = -\frac{\gamma'}{8}\left(A'\theta + \frac{C_{do}}{a} - \beta' + e'c\theta\right)$$
(3)

 $(\omega_{\beta\zeta}$  are assumed to be zero) or

$$\beta_{o} = \left[ \frac{\gamma'}{8} \left( \theta + \frac{8}{3} \theta e' - A' \right) + \beta_{p} \frac{\omega_{\beta}^{2}}{\Omega^{2}} \right] / \Omega_{\beta}^{2}$$

$$\zeta_{o} = -\frac{\gamma'}{8} \left( A'\theta + \frac{C_{do}}{a} - \beta' + e'c\theta \right) / \Omega_{\zeta}^{2} + \zeta_{p}$$
(4)

We shall use  $\beta$  and  $\zeta$  to denote the perturbed motion of the flap and the lag from the static equilibrium position. In the case of hovering ( $\mu = 0$ ), Eq. (A21) and Eq. (A22) can be reduced to

$$\beta + g_{\beta}\beta + \Omega_{\beta}^{2}\beta - X\zeta =$$

$$-2\beta\zeta - (\beta_o + \beta)\zeta^2 + \frac{\gamma'}{8}\theta\zeta^2 - \frac{\gamma'}{8}\beta\zeta \qquad (5)$$

$$\zeta \big[ 1 - (\beta_o + \beta)^2 \big] + g_\zeta \, \zeta + \Omega_\zeta^2 \zeta - Y \beta =$$

$$2\beta\beta(1+\zeta) + 2\beta_o\beta\zeta - \frac{\gamma}{8}\theta\beta\zeta + \frac{\gamma'}{8}\beta^2 \qquad (6)$$

where (  $\dot{}$  ) =  $d/d\psi, \psi=\Omega t,$  and the other parameters are defined as follows:

Presented as Paper 72-956 at the AIAA 2nd Atmospheric Flight Mechanics Conference, Palo Alto, Calif., September 11-13, 1972; submitted October 2, 1972; revision received September 25, 1973. This work was supported by Ames Directorate, U.S. Army Air Mobility Research and Development Laboratory, Ames Research Center, Moffett Field, Calif., Contract NAS2-6175. The author wishes to express his appreciation to R. H. Miller and Dr. R. A. Orimiston for their useful comments and suggestions.

Index category: Aeroelasticity and Hydroelasticity.

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$$A' = \frac{4}{1 - e} \int_{0}^{1} \lambda \xi^{2} d\xi = \frac{4}{1 - e}$$

$$B' = \frac{4}{(1 - e)^{2}} \int_{0}^{1} \lambda^{2} \xi d\xi = \frac{B}{(1 - e)^{2}}$$

$$C' = \frac{4}{(1 - e)} \int_{0}^{1} \lambda \xi d\xi = \frac{C}{1 - e}$$

$$\Omega_{\beta}^{2} = \frac{\omega_{\beta}^{2}}{\Omega^{2}} + I_{e} + 1$$

$$\Omega_{\zeta}^{2} = \frac{\omega_{\zeta}^{2}}{\Omega^{2}} + I_{e}$$

$$\gamma' = (1 - e)^{4} \gamma = (1 - e)^{4} \frac{\rho_{A} acR^{4}}{I}$$

$$I = R^{3} (1 - e)^{3} \int_{0}^{1} m \xi^{2} d\xi$$

$$I_{e} = \frac{e}{1 - e} \int_{0}^{1} m \xi^{2} d\xi$$

$$e' = e/(1 - e)$$

$$g_{\beta} = 2 \frac{\omega_{\beta}}{\Omega} \eta_{\beta} + \frac{\gamma'}{8} \left(1 + \frac{4}{3}e'\right)$$

$$g_{\zeta} = 2 \frac{\omega_{\zeta}}{\Omega} \eta_{\zeta} + \frac{\gamma'}{8} \left(A'\theta + \frac{2C_{do}}{a}\right)$$

$$X = \frac{\gamma'}{8} \left(2\theta - A' + \frac{8}{3}e'\theta\right) - 2\beta_{o}$$

$$Y = 2\beta_{o} - \frac{\gamma'}{8} \left(\theta - 2A' + \frac{4}{3}\theta e'\right)$$

in which e is the hinge offset,  $\gamma$  is the lock number, and  $\eta_{\theta}$ and  $\eta_r$  are the structural damping coefficients in flap and lag,

In Eq. (5), the first two nonlinear terms are centrifugal force terms from  $\beta(1+\zeta)$  where  $1+\zeta$  is the actual rotating speed of the blade. One may also interpret the first term as a coriolis force. The last two nonlinear terms are aerodynamic in nature from  $U_p U_T$  and  $U_T^2 \theta$ , respectively. Similarly, in Eq. (6), the nonlinear terms in the left-hand side are due to the fact that the blade has flapped up an angle  $\beta$ . Therefore, inertial force is actually  $\zeta \cos^2 \beta$ . The first two nonlinear terms in the right-hand side are coriolis terms from  $-2(1+\zeta)\beta \sin \beta$ . The last two non-linear terms are from  $U_p^2$  and  $U_p U_T \theta$ , respectively.

## III. Critical Conditions

The critical condition of the system is determined by the existence of a small amplitude undamped oscillation for the linearized equations of Eqs. (5) and Eq. (6), namely

$$\beta + g_{\beta} \beta + \Omega_{\beta}^{2} \beta - X \zeta = 0 
\zeta + g_{\zeta} \zeta + \Omega_{\zeta}^{2} \zeta - Y \beta = 0$$
(8)

To determine the critical condition, we assume

$$\beta, \zeta \sim e^{ip\psi}$$

in Eq. (8). Then we have the characteristic equation for p

$$(\Omega_{\beta}^{2} - p^{2} + ig_{\beta} p)(\Omega_{\zeta}^{2} - p^{2} + ipg_{\zeta}) + p^{2}XY = 0$$
(9)

where  $i = (-1)^{1/2}$ . The critical condition corresponds to  $p = \Omega_F$ which is called the flutter frequency. Substituting into Eq. (9) and using subscript "c" to denote the critical condition for the parameter, one obtains

$$\Omega_F^2 = \frac{g_\beta \Omega_\zeta^2 + g_{\zeta c} \Omega_\beta^2}{g_\beta + g_{\zeta c}} \tag{10}$$

$$\Omega_{F}^{2} = \frac{g_{\beta} \Omega_{\zeta}^{2} + g_{\zeta c} \Omega_{\beta}^{2}}{g_{\beta} + g_{\zeta c}}$$

$$(X_{c} Y_{c} - g_{\beta} g_{\zeta c}) \Omega_{F}^{2} = \frac{g_{\zeta c} g_{\beta} (\Omega_{\beta}^{2} - \Omega_{\zeta}^{2})^{2}}{(g_{\beta} + g_{\zeta})^{2}}$$
(11)

If all of the physical parameters such as  $\Omega_{\beta}$ ,  $\Omega_{\zeta}$ ,  $\eta_{\beta}$ , and  $\eta_{\zeta}$  are given, one can determine the critical pitch angle  $\theta_c$  from Eqs. (10) and (11). It can be shown that for  $\theta < \theta_c$ , all of the small amplitude motions will be damped and for  $\theta > \theta_c$ , there exists a branch of the solution of Eq. (8) which will grow exponentially,

To determine the critical pitch angle  $\theta_c$  for a given blade is quite complex. However, to find what the critical  $\Omega_r$  or  $\Omega_\theta$  is for a given  $\theta_c$  is very easy. For example, we can solve  $\Omega_c$  from Eqs. (10) and (11) in terms of  $\Omega_{\theta}$  and  $\theta_{c}$ ; i.e.,

$$\Omega_{\zeta}^{2} = (g_{12} + g_{15}g_{\beta}/2)/g_{12} \pm \left[ (g_{12} + g_{15}g_{\beta}/2)^{2} - g_{12}(g_{12}\Omega_{\beta}^{2} - g_{15})\Omega_{\beta}^{2}/g_{12} \right]^{1/2}$$
 (12)

in which

$$g_{12} = g_{\beta} g_{\zeta c}/(g_{\beta} + g_{\zeta c})^{2}$$
  

$$g_{15} = (X_{c} Y_{c} - g_{\beta} g_{\zeta c})/(g_{\beta} + g_{\zeta c})^{2}$$

Recall that  $g_{\beta}$  and  $g_{\zeta c}$  are the total damping coefficients in the flap and in the lag direction, X and Y are given in Eq. (7). If

$$g_{15} > 0$$

both solutions of  $\Omega_{c}$  are real. Using Eq. (12), one can easily compute the locus of the stability boundary  $\Omega_{\zeta}$  vs  $\Omega_{\beta}$  for a given  $\theta_c$ . This locus is a closed curve (Figs. 1 and 2). The region within the locus is unstable for  $\theta \ge \theta_c$ . From Eq. (12), the larger the value of  $g_{15}$ , the larger the separation of the two roots of  $\Omega_r$ . In other words the blade is less stable. For the case of no structural damping, after some algebraic manipulation, we find

$$g_{15} = \left(\frac{\gamma'}{8}\right)^2 \left\{ \left[\theta\left(1 + \frac{e}{3}\right) - A\right]^2 2(\Omega_{\beta}^2 - 1)(2 - \Omega_{\beta}^2)/\Omega_{\beta}^4 + 2\left[\theta\left(1 + \frac{e}{3}\right) - A\right] \frac{3\Omega_{\beta}^2 - 4}{\Omega_{\beta}^2} \left[\frac{4}{3}e\theta + \frac{8}{\gamma}\frac{\beta_p\omega_{\beta}^2}{\Omega^2}\right] - \left(1 + \frac{e}{3}\right)(1 - e)\frac{2C_{do}}{a} - \left(\frac{4}{3}e\theta + \frac{8\beta_p\omega_{\beta}^2}{\gamma\Omega^2}\right) \right\}$$
(13)

the effect of the hinge offset, e, and the preconing  $\beta_n$  can be easily seen. The last term in the bracket is usually small for small e and  $\beta_n$  and can be neglected. From the second term, it can be seen that if  $\Omega_{\beta}^{\ 2} > 4/3$ , the increase of e or  $\beta_p$  will increase to  $g_{15}$ ; i.e., to make the blade less stable. If  $\Omega_{\beta}^{\ 2} < 4/3$ , the increase of e or  $\beta_p$  will decrease  $g_{15}$ ; i.e., to make the blade more stable.

In practice,  $g_{\beta} \gg g_{\zeta}$ , [see Eq. (7)]. For all practical purposes, we can approximate

$$\Omega_{\rm F}^{\ 2} \cong \Omega_{\rm r}^{\ 2}$$

in Eq. (10). The solution of Eq. (8) at the critical condition is

$$\binom{\beta}{\zeta} = A \binom{u}{1} e^{i\Omega_F \psi} + A^* \binom{u^*}{1} e^{-i\Omega_F \psi} \tag{14}$$

where ( )\* denotes the complex conjugate, and

$$u = (g_{\zeta c}/Y_c) + i(\Omega_F^2 - \Omega_{\zeta}^2)/(Y_c\Omega_F)$$
  
=  $[(g_{\zeta c}X_c/g_{\beta}Y_c)]^{1/2} \exp(i \tan^{-1}[(\Omega_{\beta}^2 - \Omega_{\zeta}^2)/\Omega_F(g_{\beta} + g_{\zeta c})])$  (15)

It should be noted that  $|\beta| \sim |u| |\zeta|$  and  $|u| \sim (g_{\zeta c}/g_{\beta})^{1/2}$  which is usually of order 0.1  $\sim$  0.2; i.e., the flap motion is much smaller than that of the lag when flutter occurs. This is due to low damping in the inplane motion.

## IV. Nonlinear Solution

For  $\theta > \theta_c$ , there exists a solution of Eq. (8) which grows exponentially. In order to evaluate the amplitude of this exponentially growing solution, we must return to the nonlinear Eqs. (5) and (6). We first make a change of variable by defining

$$\beta = \alpha + u_R \zeta + (u_I/\Omega_F)\zeta \tag{16}$$

where the subscripts R and I denote the real and the imaginary part of the corresponding quantity. A substitution of Eq. (16) into Eqs. (5) and (6) yields

$$\ddot{\alpha} + g_{\beta}\dot{\alpha} + \Omega_{\beta}^{2}\alpha = -\left[\frac{u_{I}}{\Omega_{F}^{2}}\ddot{\zeta} + \left(u_{R} + \frac{g_{\beta}u_{I}}{\Omega_{F}}\right)\zeta + \left(g_{\beta}u_{R} + \Omega_{\beta}^{2}\frac{u_{I}}{\Omega_{F}} - X\right)\zeta + \Omega_{\beta}^{2}u_{R}\zeta\right] - 2\beta\zeta - (\beta + \beta_{o})\zeta^{2} + \frac{\gamma'}{8}\theta\zeta^{2} - \frac{\gamma'}{8}\beta\zeta \qquad (17)$$

$$\left(1 - \frac{Y}{Y_{c}}\frac{\Omega_{F}^{2} - \Omega_{\zeta}^{2}}{\Omega_{F}^{2}}\right)\zeta + \left(g_{\zeta} - \frac{Y}{Y_{c}}g_{\zeta c}\right)\zeta + \Omega_{\zeta}^{2}\zeta = Y\dot{\alpha} + 2\beta\beta + \left(2\beta_{o} - \frac{\gamma'}{8}\theta\right)\beta\zeta + \frac{\gamma'}{8}\beta^{2} + 2\beta\beta\zeta + \zeta(\beta_{o} + \beta)^{2} \qquad (18)$$

where  $\beta$  is in terms of  $\alpha$  and  $\zeta$  defined in Eq. (16). The right-hand side of Eqs. (17) and (18) are nonlinear terms or terms proportinal to u or Y. Both quantities u and Y are small [see Eqs. (15) and (7)]. Therefore, it is plausible to construct a series solution for  $\alpha$  and  $\zeta$  in terms of these small parameters. However, at the critical condition, the damping term in the left-hand side of Eq. (18) vanishes. The series solution will break down because of the secular solution. Therefore, besides expanding  $\alpha$  and  $\zeta$  in terms of some small parameter, we also have to introduce an expansion in the time scale  $\psi$  to suppress the secular solution whenever it appears. This is the essence of the multiple time scale technique.  $^{5,6}$ 

It can be shown that a valid expansion near the critical condition can be expressed in the following form: we are assuming all the physical parameters of the blade are fixed while  $\theta$  is in the vicinity of  $\theta_c$ ; namely

$$\theta = \theta_c + \varepsilon^2 \theta_2 \tag{19}$$

where  $\theta_2$  is a constant of order one. Usually  $|\theta_2|$  is chosen to be one or smaller and the sign  $\theta_2$  is the same as  $\theta - \theta_c$ .  $\epsilon$  is a positive small parameter where magnitude is of the order  $\theta_c$ . This implies that the quantities X, Y, and u are of order  $\epsilon$  and  $g_{\zeta}$  of order  $\theta^2$ . Using Eq. (19), all the parameters which depend on  $\theta$  can be expanded in terms of  $\epsilon$ ; i.e.,

$$g_{\zeta} = g_{\zeta c} + \varepsilon^3 g_{\zeta 3} + \dots = g_{\zeta c} + \varepsilon^2 \theta_2 (dg_{\zeta}/d\theta) + \dots$$

$$X = X_c + \varepsilon^2 \theta_2 (dX/d\theta) + \dots$$

$$Y = Y_c + \varepsilon^2 \theta_2 (dY/d\theta) + \dots$$
(20)

etc. We shall also expand the solution of  $\alpha$  and  $\zeta$  in the power series of  $\varepsilon$ . It can be shown that the proper expansions are

$$\alpha = \varepsilon^2 \alpha_4 + \varepsilon^{5/2} \alpha_5 + \cdots$$

$$\zeta = \varepsilon^{1/2} \zeta_1 + \varepsilon^3 \zeta_6 + \varepsilon^{7/2} \zeta_7 + \cdots$$
(21)

(The assumption that  $\varepsilon$  is the same order as  $\theta$ , X, Y, and u has been used to establish these expansions.) We also introduce different time scales

$$\tau_{3m} = \varepsilon^{3m} \psi; \qquad m = 0, 1, 2, \dots \tag{22}$$

for the purpose of suppressing the secular terms when they appear. Using Eq. (22), we have

$$\begin{split} \frac{\partial}{\partial \psi} &= \frac{\partial}{\partial \tau_o} + \varepsilon^3 \frac{\partial}{\partial \tau_3} \\ \frac{\partial^2}{\partial \psi^2} &= \frac{\partial^2}{\partial \tau_o^2} + 2\varepsilon^3 \frac{\partial^2}{\partial \tau_3 \partial \tau_o} \end{split}$$

Substituting Eqs. (19–22) into Eqs. (17) and (18), it is required that terms of the same order in  $\varepsilon$  satisfy Eqs. (17) and (18) separately. To write it simply, we use ( ) to denote  $\partial/\partial\tau_o$ . Then we have

 $0(\varepsilon^{1/2})$ 

$$\frac{\Omega_{\zeta}^{2}}{\Omega_{F}^{2}}(\zeta_{1}+\Omega_{F}^{2}\zeta_{1})=0 \tag{23}$$

 $0(\epsilon^2)$ 

$$\ddot{\alpha}_4 + g_{\beta} \dot{\alpha}_4 + \Omega_{\beta}^2 \alpha_4 = -2\alpha_3 \zeta_1 - \frac{\beta_o - (\gamma'/8)\theta}{\epsilon} \zeta_1^2 - \frac{\gamma'}{8} \dot{\alpha}_3 \zeta_1$$
 (24)

$$0(\varepsilon^{5/2})$$

$$\ddot{\alpha}_{5} + g_{\beta} \dot{\alpha}_{5} + \Omega_{\beta}^{2} \alpha_{5} = \theta_{2} \frac{dX}{d\theta} - a_{3} \dot{\zeta}_{1}^{2} - 2\alpha_{4} \dot{\zeta}_{1} - \frac{\gamma'}{8} \dot{\alpha}_{4} \dot{\zeta}_{1} \quad (25)$$

$$0(\varepsilon^{3})$$

$$\frac{\Omega_{\zeta}^{2}}{\Omega_{F}^{2}} (\dot{\zeta}_{6} + \Omega_{F}^{2} \dot{\zeta}_{6}) = 2\alpha_{3} \dot{\zeta}_{3} + \frac{2\beta_{o} - (\gamma'/8)\theta}{\varepsilon} \alpha_{3} \dot{\zeta}_{1} + \frac{\gamma'}{8} \dot{\alpha}_{3}^{2}$$

$$\ddot{\alpha}_{6} + g_{\beta} \dot{\alpha}_{6} + \Omega_{\beta}^{2} \alpha_{6} = -\alpha_{4} \dot{\zeta}_{1}^{2} - 2\alpha_{5} \dot{\zeta}_{1} - \frac{\gamma'}{8} \dot{\alpha}_{5} \dot{\zeta}_{1}$$

$$\frac{\Omega_{\xi}^{2/2}}{\Omega_{F}^{2}}(\zeta_{7} + \Omega_{F}^{2}\zeta_{7}) = \frac{Y}{\varepsilon}\alpha_{5} - 2\frac{\partial\zeta_{1}}{\partial\tau_{3}} + \frac{\theta_{2}(dY/d\theta)u_{I}}{\varepsilon\Omega_{F}^{2}}\zeta_{1} + \frac{\theta_{2}(dY/d\theta)u_{I}}{\varepsilon\Omega_{F}^{2}}\zeta_{1} + \frac{\theta_{2}(dY/d\theta)u_{I}}{\varepsilon}\zeta_{1} + \frac{\partial\zeta_{1}}{\varepsilon}\left(\frac{dg_{\xi}}{d\theta} - u_{R}\frac{dY}{d\theta}\right)\zeta_{1} + 2\alpha_{3}\dot{\alpha}_{4} + 2\dot{\alpha}_{3}\alpha_{4} + \frac{2\beta_{o} - (\gamma'/8)\theta}{\varepsilon}\alpha_{4}\zeta_{1} + \frac{\gamma'}{8}\dot{\alpha}_{3}\dot{\alpha}_{4} + 2\alpha_{3}\dot{\alpha}_{3}\zeta_{1} + \zeta_{1}\left(\frac{2\beta_{o}}{\varepsilon}\alpha_{4} + \alpha_{3}^{2}\right) \tag{26}$$

where

$$\alpha_3 = \frac{1}{\varepsilon} \left( u_R \zeta_1 + \frac{u_I}{\Omega_F} \zeta_1 \right)$$

The solution of Eq. (23) is simply

$$\varepsilon^{1/2}\zeta_1 = A e^{i\Omega_F t_o} + A^* e^{-i\Omega_F t_o}$$
 (27)

where A is independent of  $\tau_o$ . However, it can be a function of

Substituting Eq. (27) into Eq. (24), the right-hand side of Eq. (24) is a known function in terms of A. The solution for  $\alpha_A$  is simply

$$\varepsilon^2 \alpha_4 = 2Re \left[ a_o \frac{AA^*}{2} + a_2 A^2 e^{2i\Omega_F \tau_o} \right]$$
 (28)

in which

$$a_{o} = 2 \left[ i\Omega_{F}(u - u^{*}) + \Omega_{F}^{2} \left( \frac{\gamma'}{8} \theta - \beta_{o} - \frac{\gamma'}{8} \frac{u + u^{*}}{2} \right) \right] / \Omega_{\beta}^{2}$$

$$a_{2} = \frac{-2i\Omega_{F}u + \left( \beta_{o} - \frac{\gamma'}{8} \theta \right) \Omega_{F}^{2} + \frac{\gamma'}{8} u \Omega_{F}^{2}}{\Omega_{\beta}^{2} - 4\Omega_{F}^{2} + 2ig_{\beta}\Omega_{F}}$$
(29)

Substituting Eqs. (27) and (28) into Eq. (25), its right-hand side again becomes a known function in terms of A. We can easily solve  $\alpha_5$ ; i.e.,

$$\varepsilon^{5/2}\alpha_5 = 2Re[(\alpha_1' + a_1 A A^*) A e^{i\Omega_F \tau_o} + a_3 A^2 A^* e^{3i\Omega_F \tau_o}]$$
 (30)

n which

$$a_{1}' = (\theta - \theta_{c}) \frac{dX}{d\theta} i\Omega_{F}/(\Omega_{\beta}^{2} - \Omega_{F}^{2} + ig_{\beta}\Omega_{F})$$

$$a_{1} = \frac{\Omega_{F}^{2}(u^{*} - 2u) - 2i\Omega_{F}(a_{o} - a_{2}) - (\gamma'/4)\Omega_{F}^{2}a_{2}}{\Omega_{\beta}^{2} - \Omega_{F}^{2} + ig_{\beta}\Omega_{F}}$$

$$a_{3} = \frac{\Omega_{F}^{2}u - 2i\Omega_{F}a_{2} + (\gamma'/8)\Omega_{F}^{2}a_{2}}{\Omega_{\beta}^{2} - 9\Omega_{F}^{2} + 3ig_{\beta}\Omega_{F}}$$
(31)

We can now solve  $\alpha_6$ ,  $\zeta_6$ ,  $\zeta_7$ , etc. A substitution of Eqs. (27–31) into Eq. (26) yields

$$\frac{\Omega_{\zeta}^{2}}{\Omega_{F}^{2}}(\zeta_{7} + \Omega_{F}^{2}\zeta) = \frac{2}{\varepsilon^{7/2}}Re\left[P_{1}e^{i\Omega_{F}\tau_{o}} + P_{3}e^{3i\Omega_{F}\tau_{o}} + \cdots\right]$$
(32)

in which

$$P_{1} = -2i\Omega_{F} \varepsilon^{3} \frac{\partial A}{\partial \tau_{3}} + i\Omega_{F}(\theta - \theta_{c}) \left(\frac{dg_{\zeta}}{d\theta} - \frac{dY}{d\theta} u\right) +$$

$$iY\Omega_{F}(a_{1}' + AA^{*}a_{1})A +$$

$$\left[2i\Omega_{F}(a_{o} u + a_{2}u^{*}) + 2\Omega_{F}^{2} \left(2\beta_{o} - \frac{\gamma'}{8}\theta\right) a_{2} +$$

$$\frac{\gamma'}{2}\Omega_{F}^{2} a_{2}u^{*} + 2\Omega_{F}^{2}u^{2} - \Omega_{F}^{2}(2uu^{*} + 2\beta_{o} a_{o} +$$

$$u^{2} + 2a_{2}\beta_{o})\right]A^{2}A^{*}$$
(33)

(35)

It is clear in Eq. (32), that unless  $P_1$  is zero, the solution of  $\zeta_7$ will be in the form

$$\zeta_7 \sim \tau_o e^{i\Omega_F \tau_o}$$

which is called the secular solution. To suppress such a secular solution, we require  $P_1 = 0$ , or

$$\varepsilon^3 \frac{\partial A}{\partial \tau_3} = (\theta - \theta_c) \kappa_2 A + \kappa_3 A^2 A^* \tag{34}$$

which is the differential equation for the determination of A,

$$\kappa_{2} = \frac{1}{2} \left( \frac{dY}{d\theta} u - \frac{dg_{\zeta}}{d\theta} \right) + \frac{1}{2} \frac{Y}{X} \frac{dX}{d\theta} u$$

$$\kappa_{3} = a_{o} u + a_{2} u^{*} - i\Omega_{F} \left( 2\beta_{o} - \frac{\gamma'}{8} \theta \right) a_{2} - i\frac{\gamma'}{4} \Omega_{F} a_{2} u^{*} - i\Omega_{F} u^{2} + i\Omega_{F} \left( uu^{*} + \beta_{o} a_{o} + \frac{u^{2}}{2} + a_{2} \beta_{o} \right) \tag{35}$$

Equation (35) can be solved by expressing

$$A = \rho(\tau_3) e^{i\phi(\tau_3)}$$

where  $\rho$  and  $\phi$  are real. We now have

$$\varepsilon^{3} \frac{\partial \rho}{\partial \tau_{2}} = \left[ (\theta - \theta_{c}) \kappa_{2R} + \kappa_{3R} \rho^{2} \right] \rho \tag{36}$$

$$\varepsilon^3 \frac{\partial \phi}{\partial \tau_3} = (\theta - \theta_c) \kappa_{2I} + \kappa_{3I} \rho^2 \tag{37}$$

Equations (36) and (37) are in the same form as the set of equations obtained in Refs. 3 and 8 and also in Ref. 7 for panel flutter problems. Using the fact that  $\tau_3 = \varepsilon^3 \psi$ , see Eq. (22). The solution of Eq. (36) is simply

$$\rho = \frac{\rho_o e^{(\theta - \theta_c)\kappa_{2R}\psi}}{\left[1 + \frac{\kappa_{3R} \rho_o^2}{(\theta - \theta_c)\kappa_{2R}} (1 - e^{2(\theta - \theta_c)\kappa_{2R}\psi})\right]^{1/2}}$$

$$\phi = \phi_o + \frac{\kappa_{2I}\kappa_{3R} - \kappa_{3I}\kappa_{2R}}{\kappa_{3R}} (\theta - \theta_c)\psi + \frac{\kappa_{3I}}{\kappa_{3R}} \ln \frac{\rho}{\rho_o}$$
and  $\rho_o$  are the values of  $\rho$  and  $\phi$  at  $\tau_3 = 0$ . The

conclusions concerning the stability of the solution are the same as that in Sections IVa of Ref. 3; namely, a) if  $(\theta - \theta_c)\kappa_{2R} > 0$ (above critical conditions) and  $\kappa_{3R} > 0$ , the blade is unstable; b) if  $(\theta - \theta_c)\kappa_{2R} < 0$  (below critical condition) and  $\kappa_{3R} < 0$ , the blade is stable; c) if  $(\theta - \theta_c)\kappa_{2R} > 0$  and  $\kappa_{3R} < 0$ , there exists a limit cycle oscillation with amplitude being

$$|A|_{\mathfrak{r}_3 \to \infty} = \left[ -\frac{(\theta - \theta_c) \kappa_{2R}}{\kappa_{3R}} \right]^{1/2} \tag{39}$$

From Eqs. (7), (15), and (35), it can be shown that

$$\kappa_2 \sim \theta_c$$

and from Eqs. (7), (29), (31), and (35), it can be shown that

$$\kappa_3 \sim \theta_c^2$$

Therefore, the limit cycle amplitude is of order
$$|A|_{r_{c}\to\infty} \sim \left(\frac{\theta - \theta_{c}}{\theta_{c}}\right)^{1/2} \tag{40}$$

d) If  $(\theta - \theta_c)\kappa_{2R} < 0$  and  $\kappa_{3R} > 0$ , there exists an unstable limit cycle. The blade is stable if the disturbance in |A| is smaller than that in Eq. (39), and the blade becomes unstable if the disturbance in |A| is larger than that in Eq. (39).

## **Conclusions and Numerical Results**

It is clear that the effects of the nonlinearity are characterized by the sign and the magnitude of  $\kappa_{3R}$ , the real part of  $\kappa_3$ . In the present problem,  $\kappa_{2R}$  is positive (as can be shown). If  $\kappa_{3R} < 0$ , the blade is stable for  $\theta < \theta_c$  and has a stable limit

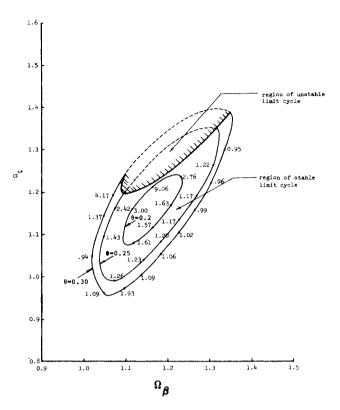


Fig. 1 Locus of critical conditions for various pitch angle  $\theta_c$  and  $\gamma = 5$ ,  $\sigma = 0.05, C_{d_p} = 0.01, a = 2\pi, \eta_{\beta} = \eta_{\zeta} = \Omega_{\beta\zeta} = \beta_p = \zeta_p = e = 0.$ 

cycle oscillation for  $\theta > \theta_c$ ; if  $\kappa_{3R} > 0$ , the blade is unstable for  $\theta > \theta_c$  and has an unstable limit cycle oscillation for  $\theta < \theta_c$ . In both cases, the limit cycle amplitude is inversely proportional to the square root of  $|\kappa_{3R}|$ . This behavior of the solution can be directly seen from Eq. (36) since the sign of  $\partial \rho/\partial \tau_3$  is the same as that of  $\kappa_{3R}$  when  $\rho$  is sufficiently large. Therefore, if  $\kappa_{3R}$  has a large positive value,  $\rho$  will grow rapidly when it is sufficiently large; i.e., the nonlinear effect is strongly destabilizing and vice versa. It is a very undesirable situation to have the blade with  $\kappa_{3R} > 0$  and hence the possibility of an unstable limit cycle. In such a case instability can always be excited if the disturbance is sufficiently large; i.e., if the initial value of  $\rho_a$  is larger than the limit cycle amplitude given in Eq. (39). In practice, large disturbances are likely to occur, due to gust or maneuver.

For a given blade,  $\kappa_{3R}$  depends upon the geometric and the structural parameters such as  $\Omega_{\beta}$ ,  $\theta_{c}$ ,  $\Omega_{\zeta}$ ,  $\beta_{p}$ ,  $\zeta_{p}$ , etc., in a complicated way. A closer examination of the terms in  $\kappa_{3}$  in Eq. (35) reveals that in the flap equation, the nonlinear term  $\beta \zeta$  usually acts to stabilize while  $\theta \zeta$  acts to destabilize the blade; the nonlinear terms due to centrifugal force are usually stabilizing if  $\Omega_r < \Omega_g$  and vice versa. However, the algebraic relation is so complex, it will be easier to examine  $\kappa_3$  numerically for a given blade. In the two cases considered in Figs. 1 and 2, when  $\Omega_r > \Omega_{\theta}$ ,  $\kappa_{3R}$  is positive or has a very small negative value; i.e., nonlinear effect is destabilized or is very weak. In the region  $\Omega_{\zeta} < \Omega_{\beta}$ ,  $\kappa_{3R}$  is generally negative, and has a larger value than that in the region  $\Omega_{\zeta} > \Omega_{\beta}$  i.e., the nonlinear effect is stabilizing and is stronger. As shown in Eq. (40), the amplitude of the blade motion grows as  $[(\theta - \theta_c)/\theta_c]^{1/2}$ , which becomes quite large as  $(\theta - \theta_c)^{1/2}$  increases. Therefore, from a practical point of view, it is not desirable to operate above the critical condition.

As shown in Sec. III, the value of the critical pitch angle  $\theta_c$  of a given blade is affected by the elastic hinge-offset, e, and the preconing,  $\beta_p$ , angle. If  $\Omega_{\beta}^2 > 4/3$ , the increase of e and  $\beta_p$  will lower  $\theta_c$  and if  $\Omega_{\beta}^2 < 4/3$ , the increase of e and  $\beta_p$  will increase  $\theta_c$ . When flutter occurs, the amplitude of lag motion is much larger than that of flap. This is due to the fact that the total damping in lag is much smaller than that in flap.

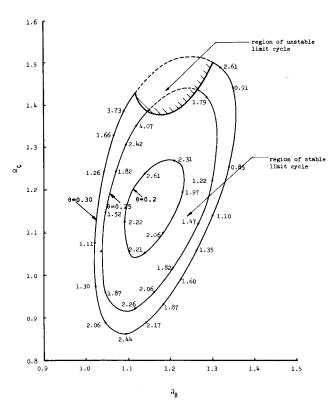


Fig. 2 Locus of critical conditions for various pitch angle  $\theta_c$  and  $\gamma=10$ ,  $\sigma=0.05,\ C_{d_o}=0.01,\ a=2\pi,\ \eta_\beta=\eta_\zeta=\Omega_{\beta\zeta}=\beta_p=\zeta_p=e=0$ .

The ratio of the amplitudes is

$$\frac{|\beta|}{|\zeta|} = |u|$$

where u is defined in Eq. (15) and usually has a value of  $0.1 \sim 0.2$ .

The solution for the case of forward flight is much more complicated and a detailed discussion can be found in Ref. 8.

Some numerical results are presented in the figures. The induced flow is approximated by

$$\lambda = \frac{\pi\sigma}{8} \left[ \left( 1 + \frac{12\theta}{\pi\sigma} - 1 \right) \right]^{1/2}$$

where  $\sigma$  is the solidity ratio.<sup>4</sup> Throughout the computation, the following parameters are used

$$\sigma = 0.05$$

$$\eta_{\beta} = \eta_{\zeta} = \Omega_{\beta\zeta} = 0$$

$$C_{do} = 0.01$$

$$a = 2\pi$$

For practical interest, we shall only compute the cases  $\Omega_{\beta} \approx 1.2$  and  $\Omega_{c} \approx 1$ .

In Figs. 1 and 2, the locus of the critical conditions for various  $\theta_c$  are presented,  $\gamma=5$  in Fig. 1 and  $\gamma=10$  in Fig. 2. These locus are similar to that of Fig. 5 of Ref. 4. The region within the locus is unstable according to linear aeroelasticity theory, if the blade is operating at a pitch angle greater than the corresponding  $\theta_c$ . For  $\gamma=10$ , such region is much larger than that for  $\gamma=5$  for the same  $\theta_c$ ; i.e., for larger  $\gamma$ , the blade is less stable. The numbers given on this locus are the values  $|\kappa_{2R}/\kappa_{3R}|^{1/2}$  for the corresponding parameters at  $\theta=\theta_c$ . The solid line is a portion of the locus where  $\kappa_{3R}<0$  and the dotted line is  $\kappa_{3R}>0$ . That is, over the solid line portion of the locus, there exists a stable limit cycle of amplitude  $2(\theta-\theta_c)^{1/2}|\kappa_{2R}/\kappa_{3R}|^{1/2}$  for lag motion and  $2(\theta-\theta_c)^{1/2}|\kappa_{2R}/\kappa_{3R}|^{1/2}|\mu|$  for the flap when  $\theta>\theta_c$ .  $|\mu|$  is usually  $0.1\sim0.2$  because of low aerodynamic damping in lag. The blade is unconditionally stable if  $\theta<\theta_c$ .

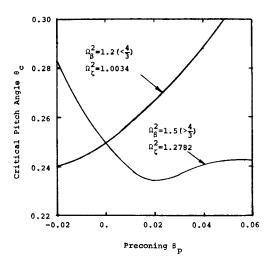


Fig. 3 The critical pitch angle  $\theta_c$  vs the preconing  $\beta_p$ ,  $\gamma=5$ ,  $\sigma=0.05$ ,  $C_{d_o}=0.01$ ,  $a=2\pi$ ,  $\eta_{\beta}=\eta_{\zeta}=\Omega_{\beta\zeta}=e=0$ .

Over the dotted portion of the locus, the blade is unconditionally unstable if  $\theta > \theta_c$  and conditionally stable if  $\theta < \theta_c$ ; i.e., if the disturbance in |A| is smaller than  $|\kappa_{2R}/\kappa_{3R}|^{1/2}$ , it will die out and if it is larger than  $|\kappa_{2R}/\kappa_{3R}|^{1/2}$ , it will grow. In these two cases, the limit cycle amplitude can be quite large for  $\theta > \theta_c$ , because  $|\kappa_{2R}/\kappa_{3R}|$  are quite large; i.e., the nonlinear effect is weak. Therefore, it is not desirable to operate above the critical condition.

Figure 3 shows the effect of the preconing angle  $\beta_p$  as predicted in Section III. For  $\Omega_{\beta}^2 = 1.2(<4/3)$ , a positive preconing increases  $\theta_c$  and for  $\Omega_{\beta}^2 = 1.5(>4/3)$ , a positive preconing decreases  $\theta_c$ .

Figures 4 and 5 show the lag responses of the blade in hovering for a given initial disturbance, obtained by the predictor-corrector method of numerical integration. In both cases,  $\Omega_{\rm g}=1.09544,\ \gamma=5,\ \theta=0.3,\ {\rm and}\ \theta_{\rm c}=0.25.\ \Omega_{\rm c}=1.0017$  for Fig. 4 and  $\Omega_{\rm c}=1.19747$  for Fig. 5, while the numerical value of  $\kappa_{2\rm R}$  is 0.0118 and the numerical values of  $\kappa_{3\rm R}$  are -0.0225 and 0.0042, respectively, for both cases. It is clear that the motion in Fig. 4 approaches a limit cycle oscillation while the amplitude in Fig. 5 grows steadily.

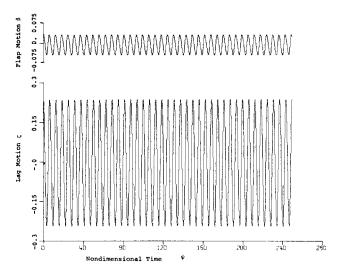


Fig. 4 Flap and lag motion in hovering for pitch angle above critical condition.  $\theta=0.29, \gamma=5, \sigma=0.05, C_{d_c}=0.01, a=2\pi, \eta_{\beta}=\eta_{\zeta}=\Omega_{\beta\zeta}=\beta_p=\zeta_p=e=0, \ \Omega_{\beta}=1.09544, \ \Omega=1.0017.$  In this case  $\theta_c=0.25, \Omega_F=1.00418.$  Limit cycle oscillation exists.

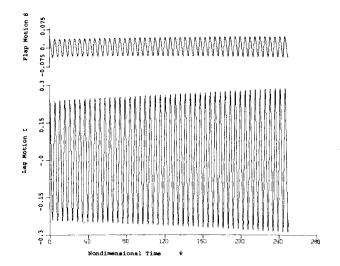


Fig. 5 Flap and lag motion in hovering for pitch angle above critical condition,  $\theta=0.25,\ \gamma=5,\ \sigma=0.05,\ C_{d_a}=0.01,\ a=2\pi,\ \eta_{\beta}=\eta_{\zeta}=\Omega_{\beta\zeta}=\Omega_{\beta\zeta}=\beta_p=\zeta_p=e=0,\ \Omega_{\beta}=1.09544,\ \Omega_{\zeta}=1.19747.$  In this case  $\theta=0.25,\ \Omega_{F}=1.195.$  No stable limit cycle exists.

Figure 6 plots the limit cycle amplitude of the blade with  $\gamma=5$ ,  $\Omega_{\beta}=1.09544$ , and  $\Omega_{\zeta}=1.0017$ . The results are in good agreement with those obtained directly by direct numerical integration.

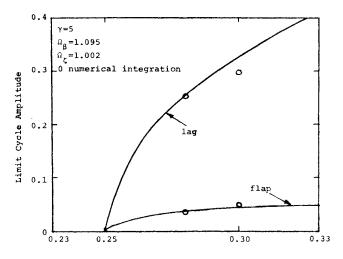
### Appendix A Derivation of the Flap-Lag Equations

In deriving the equation of motion of the blade in flaplag, we define the flap,  $\beta$ , as the instantaneous angle between the blade and the hub plane, the lag  $\zeta$  as the instantaneous angle between the blade and the feathering axis, and  $\theta$  as the pitch angle about the feathering axis (Fig. 7). Two coordinates are used, namely,  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  rotating with the blade about the shaft and  $(\mathbf{I}, \mathbf{J}, \mathbf{K})$  rigidly attached to the blade. The relation between the two sets of base vectors are

$$\mathbf{i} = \cos \beta \mathbf{I} - \sin \beta \sin \theta \mathbf{J} - \sin \beta \cos \theta \mathbf{K}$$

$$\mathbf{j} = \cos \theta \mathbf{J} - \sin \theta \mathbf{K}$$

$$\mathbf{k} = \sin \beta \mathbf{I} + \cos \beta \sin \theta \mathbf{J} + \cos \beta \cos \theta \mathbf{K}$$
(A1)



Pitch Angle 6

Fig. 6 Limit cycle amplitude vs pitch angle  $\gamma=5, \sigma=0.05, C_{d_b}=0.01,$   $a=2\pi, \eta_\beta=\eta_\zeta=\Omega_{g_\zeta}\beta_p=\zeta_p=e=0. \Omega_\beta=1.0954, \Omega_\zeta=1.0017.$ 

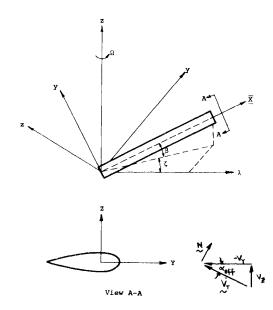


Fig. 7 Geometry and coordinates.

Let R be the distances from the tip of the blade to the axis of rotation at the undeformed state  $(\beta = 0, \zeta = 0)$ . Consider the point at a distance of  $R(1-e)\xi$  from the hinge offset which is located at a distance Re from the center of the hub. The position, the velocity and the acceleration of the point are

$$\mathbf{r} = Re\,\mathbf{i}' + R(1 - e)\xi(\cos\beta\mathbf{i} + \sin\beta\mathbf{k}) \tag{A2}$$

$$\mathbf{V} = \Omega Re \,\mathbf{j}' + \Omega R(1 - e)\xi \left[ -\beta \sin \beta \mathbf{i} + (1 + \dot{\zeta})\cos \beta \mathbf{j} + \beta \cos \beta \mathbf{k} \right]$$
(A3)

$$\mathbf{a} = -\Omega^2 Re \,\mathbf{i}' + \Omega^2 (1 - e) R\xi \{ -\left[\beta \sin \beta + \beta^2 \cos \beta + (1 + \zeta)^2 \cos \beta\right] \mathbf{i} + \left[\zeta \cos \beta - 2(1 + \zeta)\beta \sin \beta\right] \mathbf{j} + \left[\beta \cos \beta - \beta^2 \sin \beta\right] \mathbf{k} \}$$
(A4)

in which ( ) denotes differential with respect to  $\psi$ ,  $\psi = \Omega t$  where  $\Omega$  being the rotating speed of the shaft itself and

$$i' = \cos \zeta i - \sin \zeta j$$
  

$$j' = \sin \zeta i + \cos \zeta j$$
(A5)

Let m be the mass per unit length, then the moment of inertia at the hinge offset is

$$\mathbf{M}_{I} = -\int_{0}^{1} m(\mathbf{r} - Re\,\mathbf{i}') \times \mathbf{a}R(1 - e) \, d\xi =$$

$$-\Omega^{2} II_{e} \left[ -\sin\beta \left( \sin\zeta\mathbf{i} + \cos\zeta\mathbf{j} \right) + \sin\zeta\cos\beta\mathbf{k} \right] +$$

$$\Omega^{2} I\left\{ \left[ \zeta \cos\beta - 2(1 + \zeta)\beta\sin\beta \right] \times \left( \sin\beta\mathbf{i} - \cos\beta\mathbf{k} \right) + \right.$$

$$\left[ \beta + (1 + \zeta)^{2}\sin\beta\cos\beta \right] \mathbf{j} \right\}$$
(A6)

where

$$I = R^{3}(1-e)^{3} \int_{0}^{1} m\xi^{2} d\xi$$

$$II_{e} = R^{3}(1-e)^{2}e \int_{0}^{1} m\xi d\xi$$
(A7)

In the case of a uniform mass distribution

$$I_e = (3/2)e/(1-e)$$
 (A8)

The flow velocity relative to the blade is

$$\mathbf{V}_{A} = \mu R\Omega \left[\cos\left(\psi + \zeta\right)\mathbf{i} - \sin\left(\psi + \zeta\right)\mathbf{j}\right] + \mathbf{V}_{\text{Ind}} - \mathbf{V}$$
 (A9)

in which  $V_{lnd}$  is the induced velocity and v is given in Eq. (A3). To evaluate the lift and the drag, it is more convenient to express  $V_A$  in terms of (I, J, K), the base vector moving together with the blade, i.e.

$$\mathbf{V}_{A} = V_{x}\mathbf{I} + V_{y}\mathbf{J} + V_{z}\mathbf{K} \tag{A10}$$

It is clear that  $V_x$  is the velocity component in the axis direction and  $V_y$  and  $V_z$  are in the cross section plane of the blade. We assume

$$V_{Ind} = -\Omega R \lambda K \tag{A11}$$

using Eq. (A1) we find

We shall assume that quasi-steady two-dimensional aerodynamics theory holds for each cross section, i.e., the aerodynamics load vector can be written as

$$\mathbf{L}_{A} = \frac{1}{2}\rho_{A} ac \left| V_{T} \right|^{2} \left( \alpha_{\text{eff}} \mathbf{N} + \frac{C_{d_{o}}}{a} \frac{\mathbf{V}_{T}}{\left| \mathbf{V}_{T} \right|} \right)$$
 (A13)

where  $\rho_A$  is the air density, "a" lift slope, c chord length,  $C_{d_a}$  profile drag coefficient

$$\mathbf{V}_{T} = V_{\nu} \mathbf{J} + V_{z} \mathbf{K} \tag{A14}$$

 $\alpha_{eff}$  is the effective angle of attack, i.e., (Fig. 1)

$$\alpha_{\text{eff}} = \tan^{-1}(V_z/V_y) \cong (V_z/V_y)$$
 (A15)

and N is a unit vector normal to  $V_T$  and in the plane of the blade cross section defined by

$$\mathbf{N} = (\mathbf{I} \times \mathbf{V}_T / |\mathbf{Y}_T|) (V_y / |V_y|) \tag{A16}$$

If  $V_y > 0$ , i.e., the flow is from the trail edge to the leading edge; this will be called the reverse flow. In our present consideration, we assume that  $V_y < 0$ , and  $\alpha_{\rm eff}^2 \ll 1$ ,  $C_{d_0}/a \ll 1$ . A substitution of Eqs. (A14–A16) into Eq. (A13) yields

$$\mathbf{L}_{A} = (L_{\mathbf{x}})_{A} \mathbf{i} + (L_{\mathbf{y}})_{A} \mathbf{j} + (L_{\mathbf{z}})_{A} \mathbf{k}$$
 (A17)

where

$$\begin{split} (L_{y})_{A} &= \frac{1}{2}\rho_{A} \, ac \left\{ \left[ \left. V_{z}^{2} + \frac{C_{d_{\theta}}}{a} \middle| V_{y} \middle| V_{y} \right] \cos \theta - \right. \\ &\left. \left[ \frac{C_{d_{\theta}}}{a} \middle| V_{y} \middle| V_{z} - V_{z} V_{y} \right] \sin \theta \right\} \end{split}$$

$$(L_x)_A = -\frac{1}{2}\rho_A ac \left\{ \left[ V_z^2 + \frac{C_{d_0}}{a} |V_y| V_y \right] \sin \theta + \left[ \frac{C_{d_0}}{a} |V_y| V_z - V_z V_y \right] \cos \theta \right\} \sin \beta$$
 (A18)

 $(L_z)_A = -(L_x)_A (\cos \beta / \sin \beta)$ 

The aerodynamic moment at the hub (z-axis) is

$$\mathbf{M}_{A} = \int_{0}^{1} (\mathbf{r} - Re\,\mathbf{i}') \times \mathbf{L}_{A} R(1 - e) \,d\xi \tag{A19}$$

The equations of motion of the blade are

$$(\mathbf{M}_{I} + \mathbf{M}_{A}) \cdot \mathbf{j} + 2I\omega_{\beta}\eta_{\beta}\beta + I\omega_{\beta}^{2}(\beta - \beta_{p}) + I\omega_{\beta\zeta}(\zeta - \zeta_{p}) = 0 \quad (A20)$$
in flap and

 $(\mathbf{M}_I + \mathbf{M}_A) \cdot \mathbf{k} + 2I\omega_\zeta \eta_\zeta \zeta + I\omega_\zeta^2(\zeta - \zeta_p) + I\omega_{\beta\zeta}(\beta - \beta_p) = 0$  (A21) in lag, where  $\beta_p$  and  $\zeta_p$  are the preconing and prelagging angles, and  $\omega_{\beta}$ ,  $\omega_{\zeta}$  are the bending frequencies in the flap and in the lag direction, and  $\omega_{\beta\zeta}$  is due to elastic coupling. (Note that, depending on the construction of the blade,  $\omega_{\beta}$ ,  $\omega_{\zeta}$ ,  $\omega_{\beta\zeta}$  can be a function of the pitch angle.)

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